DYNAMIC INTERACTION OF SYSTEMS OF CRACKS UNDER ANTIPLANE DEFORMATION CONDITIONS

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Problems of the dynamic effect on an isolated crack located in an infinite elastic body were solved in [1-4]. It is interesting to obtain the solution of dynamic problems corresponding to a more complex geometry, and to clarify the influence of the presence of adjacent cracks, systems of cracks, and the body boundaries.

INTRODUCTION

The mathematical description of an elastic body is substantially simpler for antiplane deformation than for plane deformation but it accurately reflects the characteristic features of the phenomenon. In this case, exact solutions of the limit problems are obtained successfully when the crack length is much greater than either the spacing between them or the spacing to the half-space boundary. The method of solution used is carried over to the case of plane deformation without special difficulties.

1. SYSTEM OF PARALLEL CRACKS

An elastic isotropic space containing an infinite number of cracks of lengths $2l_0$ and 2L which are in parallel and separated by the spacing 2h is considered. Under antiplane deformation conditions the single nonzero component of the displacement vector is $w = w(x, y, t)$. Let us introduce the dimensionless variables

$$\langle L,\ h,\ x,\ y,\ w\rangle' = \langle L,\ h,\ x,\ y,\ w\rangle / l_0;\ \tau' = \tau/\mu;\ t' = tc/l_0, \tag{1.1}$$

where c is the velocity of the transverse waves; μ is the shear modulus. We henceforth omit the primes to simplify the writing. Then the equation of motion of an isotropic elastic body and the nonzero components of the stress tensor are

$$\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 - \partial^2 w/\partial t^2 = 0;\ \tau_{yz} = \partial w/\partial y;\ \tau_{xz} = \partial w/\partial x. \tag{1.2}$$

Let us assume that $w = 0$ everywhere for $t < 0$, while $w = 0$ and $\dot{w} = 0$ for $t = 0$ and the applied stresses τ_{yz} are even functions of x. By virtue of symmetry of the problem relative to any line passing through one of the cracks, we shall limit ourselves to the examination of an infinite strip $-h < y < +h$. The boundary conditions for $t > 0$ have the form

$$\tau_{yz} = \begin{cases} -p\,(x,t),\ y = h,\ |x| < 1, \\ \mp p\,(x,t),\ y = -h,\ |x| < L; \end{cases}$$
$$w = 0,\ y = h,\ |x| > 1 \text{ and } y = -h,\ |x| > L. \tag{1.3}$$

Performing a Laplace integral transform in t and a Fourier cosine transform in x, we obtain for the equation of motion (1.2)

$$\partial^2\overline{w}/\partial y^2 - (s^2 + p^2)\overline{w} = 0,\ \overline{w} = w(s,\ y,\ p).$$

The general solution of this equation is

$$w(s,\ y,\ p) = A(s,\ p)\operatorname{sh}\alpha y + B(s,\ p)\operatorname{ch}\alpha y,\ \alpha = \sqrt{s^2 + p^2}.$$

Substituting it into (1.3), we obtain a system of equations to determine $A(s, p)$ and $B(s, p)$:

$$2/\pi \int_0^\infty [A(s,\ p)\operatorname{sh}\alpha h + B(s,\ p)\operatorname{ch}\alpha h]\cos(sx)\,ds = 0,\ x > 1; \tag{1.4}$$

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$$2/\pi \int_0^\infty [-A(s, p)\,\mathrm{sh}\,\alpha h + B(s, p)\,\mathrm{ch}\,\alpha h] \cos(sx)\, ds = 0,\ x > L;$$

$$2/\pi \int_0^\infty \alpha\, [A(s, p)\,\mathrm{ch}\,\alpha h + B(s, p)\,\mathrm{sh}\,\alpha h] \cos(sx)\, ds = -P(x, p),\ 0 < x < 1;$$

$$2/\pi \int_0^\infty \alpha\, [A(s, p)\,\mathrm{ch}\,\alpha h - B(s, p)\,\mathrm{sh}\,\alpha h] \cos(sx)\, ds = \mp P(x, p),\ 0 < x < L,$$

where

$$P(x, p) = \int_0^\infty p(x, t)\, e^{-pt}\, dt.$$

It is known from the theory of cracks that the displacements at the nose of the crack should behave as follows:

$$w(x,\ h,\ p) \sim (1 - x^2)^{1/2},\ x = 1 - \varepsilon;$$
$$w(x,\ -h,\ p) \sim (L^2 - x^2)^{1/2},\ x = L - \varepsilon;\ \varepsilon \ll 1.$$

Let us introduce the two functions $\varphi(t, p)$ and $\psi(t, p)$, defined with respect to t in the intervals [0, 1] and [0, L], respectively, by the equalities

$$w(x, h, p) = \int_x^1 \frac{t\varphi(t, p)}{\sqrt{t^2 - x^2}}\, dt;\ w(x, -h, p) = \int_x^L \frac{t\psi(t, p)}{\sqrt{t^2 - x^2}}\, dt. \tag{1.5}$$

Using (1.5), the first two equations in (1.4) can be written as

$$A(s, p)\,\mathrm{sh}\,\alpha h + B(s, p)\,\mathrm{ch}\,\alpha h = \pi/2 \int_0^1 t\varphi(t, p)\, J_0(st)\, dt = \pi\Phi/2;$$
$$-A(s, p)\,\mathrm{sh}\,\alpha h + B(s, p)\,\mathrm{ch}\,\alpha h = \pi/2 \int_0^L t\psi(t, p)\, J_0(st)\, dt = \pi\Psi/2. \tag{1.6}$$

We hence find A(s, p) and B(s, p):

$$A(s, p) = \pi/4 \cdot [\Phi - \Psi]\,\mathrm{sh}^{-1}\alpha h;\ B(s, p) = \pi/4 \cdot [\Phi + \Psi]\,\mathrm{ch}^{-1}\alpha h. \tag{1.7}$$

Integrating the third and fourth equations from (1.4) with respect to x between 0 and x, and using (1.6) and (1.7), we obtain

$$\int_0^\infty \frac{\alpha F}{2s} \Phi \sin(sx)\, ds - \int_0^\infty \frac{\alpha G}{2s} \Psi \sin(sx)\, ds = -\int_0^x P(x, p)\, dx;$$
$$\int_0^\infty \frac{\alpha G}{2s} \Phi \sin(sx)\, ds - \int_0^\infty \frac{\alpha F}{2s} \Psi \sin(sx)\, ds = \mp \int_0^x P(x, p)\, dx, \tag{1.8}$$

where $F = \coth\alpha h + \tanh\alpha h$; $G = \coth\alpha h - \tanh\alpha h$. The first equation is valid for $0 \le x \le 1$, and the second, for $0 \le x \le L$.

Let us reduce (1.8) to two Fredholm integral equations of the second kind. Let us show this by the example of the first equation from (1.8). Let us introduce the function g(s, p) by means of the equation

$$\alpha F/(2s) = 1 + g(s,\ p),\ g(s) \sim O(s^{-2}) \quad \text{as} \quad s \to \infty. \tag{1.9}$$

Substituting it into the first equation from (1.8), we obtain an Abel integral equation:

$$\int_0^x \frac{t\varphi(t, p)}{\sqrt{x^2 - t^2}}\, dt = H(x),\ 0 \leqslant x \leqslant 1;$$

$$H(x) = -\int_0^x P(x, p)\, dx - \int_0^1 t\varphi(t, p)\, dt \int_0^\infty g(s, p)\, J_0(st) \sin(sx)\, ds + \int_0^L t\psi(t,\ p)\, dt \int_0^\infty \frac{\alpha G}{2s} J_0(st) \sin(sx)\, ds,$$

whose solution is $\varphi(t, p) = 2/\pi \int_0^t \frac{H'(x)\, dx}{\sqrt{t^2 - x^2}}$. Substituting H'(x) here and integrating with respect to x, we obtain

$$\varphi_1(t, p) + \int_0^1 \varphi_1(\tau, p) K_2(\tau, t) d\tau - \int_0^L \psi_1(\tau, p) K_1(\tau, t) d\tau = -\sqrt{t}, \quad 0 \leqslant t \leqslant 1. \tag{1.10}$$

We obtain the following completely analogously for the second equation from (1.8) for $0 \le t \le L$:

$$\psi_1(t, p) + \int_0^L \psi_1(\tau, p) K_2(\tau, t) d\tau - \int_0^1 \varphi_1(\tau, p) K_1(\tau, t) d\tau = \pm \sqrt{t}, \tag{1.11}$$

where

$$\begin{bmatrix} \varphi_1(t, p) \\ \psi_1(t, p) \end{bmatrix} = \sqrt{t} \begin{bmatrix} \varphi(t, p) \\ \psi(t, p) \end{bmatrix} \left[2/\pi \int_0^t \frac{P(x, p)}{\sqrt{t^2 - x^2}} dx \right]^{-1};$$

$$K_1(\tau, t) = \sqrt{\tau t} \int_0^\infty \alpha G/2 \cdot J_0(st) J_0(s\tau) ds; \tag{1.12}$$

$$K_2(\tau, t) = \sqrt{\tau t} \int_0^\infty g(s, p) s J_0(st) J_0(s\tau) ds.$$

In the particular case $L = 1$, Eqs. (1.10) and (1.11) reduce for identical signs in the right sides to

$$\varphi_1(t, p) + \int_0^1 \varphi_1(\tau, p) K_3(\tau, t) d\tau = -\sqrt{t}, \tag{1.13}$$

$$K_3(\tau, t) = \sqrt{\tau t} \int_0^\infty [\alpha \operatorname{th} \alpha h - s] J_0(st) J_0(s\tau) ds,$$

and for different signs in the right sides, to

$$\varphi_1(t, p) + \int_0^1 \varphi_1(\tau, p) K_4(\tau, t) d\tau = -\sqrt{t}, \tag{1.14}$$

$$K_4(\tau, t) = \sqrt{\tau t} \int_0^\infty [\alpha \operatorname{cth} \alpha h - s] J_0(st) J_0(s\tau) ds.$$

These equations correspond to the problem of a central crack in a layer of thickness 2h, whose boundaries are free [$\tau_{yz} = 0$ (1.13)] and fastened [$w = 0$ (1.14)].

The main characteristic of the problems of the theory of cracks is the coefficient of stress intensity K at the nose of the crack for a singularity on the order of $(\Delta x)^{-1/2}$ ($L \ge 1$, $\Delta x \ll 1$). Let us examine the expression for $\tau_{yz}(x, \pm h, p)$, let us show that they have a singularity of the needed order, and let us find the coefficients for this singularity:

$$\tau_{yz}(x, h, p) = \int_0^\infty \alpha F/2 \cdot \Phi \cos(sx) ds - \int_0^\infty \alpha G/2 \cdot \Psi \cos(sx) ds,$$

$$\tau_{yz}(x, -h, p) = -\int_0^\infty \alpha F/2 \cdot \Psi \cos(sx) ds + \int_0^\infty \alpha G/2 \cdot \Phi \cos(sx) ds.$$

Integrating by parts in (1.6), we obtain

$$\Phi = 1/s \cdot [\varphi(1, p) J_1(s) - \int_0^1 \varphi'(t, p) t J_1(st) dt];$$

$$\Psi = 1/s \cdot \left[\psi(L, p) L J_1(sL) - \int_0^L \psi'(t, p) t J_1(st) dt \right].$$

The second members in the expressions for τ_{yz} evidently have no singularities. Taking the nonintegral parts in the expressions for Φ and Ψ and taking account of (1.9), we write just the terms yielding the singularity:

$$\tau_{yz}(x, h, p) = \int_0^\infty \varphi(1, p) J_1(s) \cos(sx) ds + \dots,$$

$$\tau_{yz}(x,-h,p)=-\int_0^\infty \psi(L,p)LJ_1(Ls)\cos(sx)\,ds+\ldots.$$

Using the known formula [5]

$$\int_0^\infty J_1(Ls)\cos(sx)\,ds=-\frac{L}{\sqrt{x^2-L^2}\,[x+\sqrt{x^2-L^2}]},$$

the coefficients for the singularities at the heads of the cracks can be written down:

$$\begin{aligned}\tau_{yz}(l_0+\Delta x,h,p)&\simeq-\frac{\varphi(1,p)}{\sqrt{2}}\sqrt{\frac{l_0}{\Delta x}}=\frac{K_1(p)}{\sqrt{\Delta x}}=-\frac{P_1\varphi_1(1,p)}{\sqrt{2}}\sqrt{l_0}(\Delta x)^{-1/2},\\ \tau_{yz}(L+\Delta x,-h,p)&\simeq\frac{\psi(L,p)}{\sqrt{2}}\sqrt{\frac{Ll_0}{\Delta x}}=\frac{K_2(p)}{\sqrt{\Delta x}}=\frac{P_L\psi_1(L,p)}{\sqrt{2}}\sqrt{l_0}(\Delta x)^{-1/2},\end{aligned}\tag{1.15}$$

where

$$P_1=2/\pi\int_0^1\frac{P(x,p)}{\sqrt{1-x^2}}dx;\quad P_L=2/\pi\int_0^L\frac{P(x,p)}{\sqrt{L^2-x^2}}dx.$$

Equations (1.10)-(1.14) were considered numerically. The kernels of the equations $K_i(\tau,t)$ ($i=2,3,4$) can be written in a form more convenient for machine calculations:

$$K_i(\tau,t)=p^2\sqrt{\tau t}\left\{1/2\begin{bmatrix}I_0\left(\frac{\tau p}{2}\right)K_0\left(\frac{tp}{2}\right)\\ I_0\left(\frac{tp}{2}\right)K_0\left(\frac{\tau p}{2}\right)\end{bmatrix}_{t<\tau}^{\tau<t}+\int_0^\infty\omega_i(\xi)J_0(\xi pt)J_0(\xi p\tau)\,d\xi\right\},\tag{1.16}$$

$$\omega_2(\xi)=\sqrt{1+\xi^2}/2\cdot[\mathrm{cth}\,ph\sqrt{1+\xi^2}+\mathrm{th}\,ph\sqrt{1+\xi^2}]-\xi-2\xi/(4\xi^2+1),$$

$$\omega_3(\xi)=\sqrt{1+\xi^2}\,\mathrm{th}\,ph\sqrt{1+\xi^2}-\xi-2\xi/(4\xi^2+1),$$

$$\omega_4(\xi)=\sqrt{1+\xi^2}\,\mathrm{cth}\,ph\sqrt{1+\xi^2}-\xi-2\xi/(4\xi^2+1),$$

$$\omega_i(\xi)\sim O(\xi^{-5})\quad\text{as}\quad\xi\to\infty.$$

Here $I_0(x)$ and $K_0(x)$ are cylindrical functions of imaginary argument. It was assumed that $p(x,t)=p_0$ throughout in the computations.

A method to find the inverse Laplace transform numerically, which is elucidated in [6], was used to construct the dependences $K_i(t)$ ($i=1, 2$) by means of (1.15).

As an illustration, results of a computation of (1.10)-(1.12) on an electronic computer and the subsequent numerical inversion of the Laplace transform are represented in Fig. 1. The dashed curves 1, 2 show the values of $K_1(t)/(p_0\sqrt{l_0})$ and the continuous curves show $K_2(t)/(p_0\sqrt{l_0})$. The curves have been constructed for $L/l_0=2$, where curves 1 correspond to the ratio $l_0/h=1$, and curves 2, to the ratio $l_0/h=2$. The upper curves correspond to stresses of identical sign acting on the cracks and the lower curves, to stresses of opposite signs. In this latter case, the screening effect of a long crack is especially graphic – its presence results in an abrupt drop in the value of the stress-intensity coefficient at the nose of a short crack. The solid lines in Fig. 2 exhibit the time dependence $K(t)/(p_0\sqrt{l_0})$ for $l_0/h=1$ (curves 1) and $l_0/h=2$ (curves 2). The upper curves hence correspond to free layer boundaries, i.e., the solution of (1.13), and the lower, to rigidly framed layer boundaries, i.e., the solution of (1.14).

The static solution obtained from (1.10)-(1.12) as a result of passing to the limit as $p\to 0$, which corresponds to $t\to\infty$, is shown in Fig. 3. The solid lines show the dependence of the ratio $K_2/(p_0\sqrt{l_0})$, on the quantity $K_1(p_0\sqrt{l_0})$. Curves 1-5 correspond to the values $l_0/h=1.0$; 0.8; 0.6; 0.4; 0.2, respectively. The upper curves correspond to stresses with the same sign acting on the cracks and the lower, to stresses of opposite signs. The dependences of the stress-intensity coefficients $K/(p_0\sqrt{l_0})$ of the static problem on the quantity h/l_0, obtained as a result of a numerical computation of (1.13) and (1.14) as $p\to 0$, are illustrated in Fig. 4. The upper curve corresponds to the condition $\tau_{yz}=0$ on the layer boundaries and the lower curve corresponds to the boundary condition $w=0$. The solutions constructed agree with the exact solutions of the corresponding static problems.

2. A CRACK PARALLEL TO THE HALF-SPACE BOUNDARY

An elastic isotropic half-space $y\geq-h$ containing an isolated crack of length 2 and located at $y=0$, $|x|<1$, is considered. We assume that at $t>0$ the stress $\tau_{yz}=\pm p(x,t)$ acts, respectively, on the crack at the upper

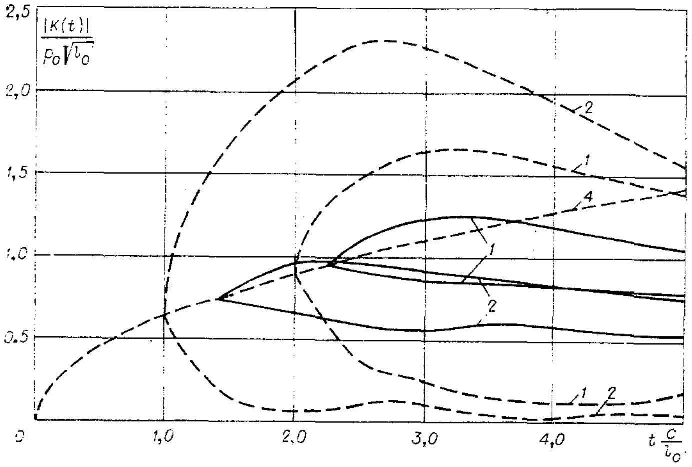

Fig. 1

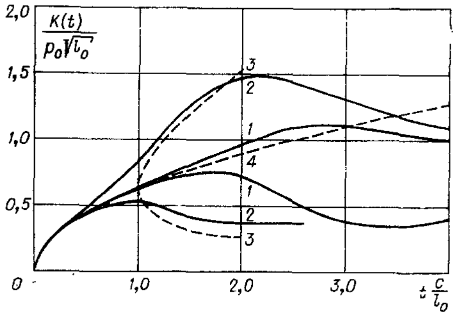

Fig. 2

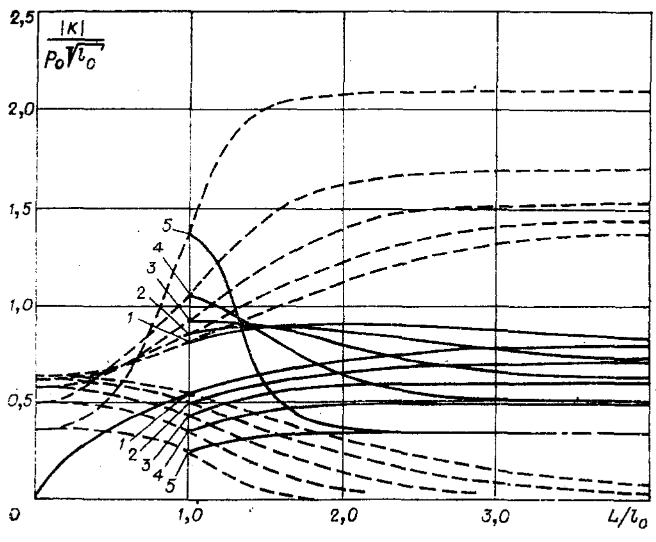

Fig. 3

and lower edges of the slit. Let us divide the domain under consideration into two. The first is the infinite strip $-h < y < 0$. The quantities referring to it will have the subscript 1. The second domain, which has the subscript 2, is the half-plane $y > 0$. Then the boundary conditions for $t > 0$ are the following:

$$w_{(1)} = 0 \text{ for } y = -h,\ |x| < \infty;\ \tau_{yz} = \mp p\,(x,\ t) \text{ for } y = 0,\ |x| < 1;$$
$$w_{(1)} - w_{(2)} = 0 \text{ for } y = 0,\ |x| > 1;\ \tau_{(1)yz} - \tau_{(2)yz} = 0 \text{ for } y = 0,\ |x| < \infty. \tag{2.1}$$

We can take

$$\tau_{(1)yz} = 0 \text{ for } y = -h,\ |x| < \infty \tag{2.2}$$

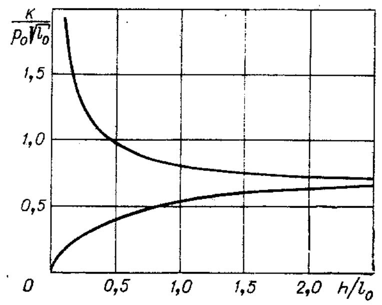

Fig. 4

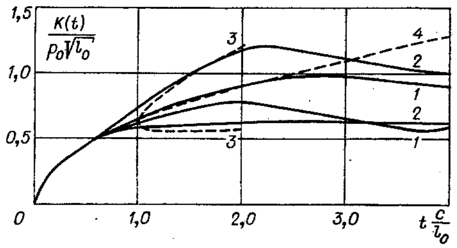

Fig. 5

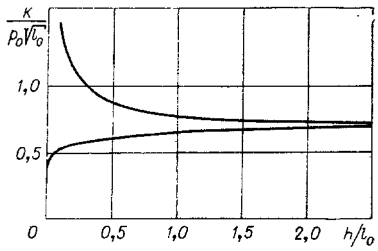

Fig. 6

in place of the first boundary condition. The general solutions of the equations of motion in the appropriate domains are

$$w_{(1)}(s,\ y,\ p) = A_1(s,\ p)\,\mathrm{sh}\,\alpha y + B_1(s,\ p)\,\mathrm{ch}\,\alpha y;$$
$$w_{(2)}(s,\ y,\ p) = A_2(s,\ p)e^{-\alpha y}.$$

Substituting these into the boundary conditions (2.1), we obtain the system of equations

$$\int_0^\infty B(s,\ p)\cos(sx)\,ds = 0, \quad |x| > 1;$$
$$\int_0^\infty B(s,\ p)\,2\alpha/[1 + \mathrm{th}\,\alpha h]\cdot\cos(sx)\,ds = \pi/2\cdot P(x,\ p),\ |x| < 1,$$

where $2B(s,\ p) = A_1(s,\ p)[1 + \mathrm{th}\,\alpha h]$.

Exactly as before, let us introduce the function $\varphi(t,\ p)$, defined in the interval [0, 1] with respect to t by the equality

$$w_{(1)}(x,\ 0,\ p) - w_{(2)}(x,\ 0,\ p) = \int_x^1 \frac{t\varphi(t,\ p)}{\sqrt{t^2 - x^2}}\,dt.$$

Proceeding analogously to the above, we obtain a Fredholm integral equation of the second kind:

$$\varphi_1(t,\ p) + \int_0^1 \varphi_1(\tau,\ p)\,K_1(\tau,\ t)\,d\tau = -\sqrt{t},\ 0 \leqslant t \leqslant 1, \tag{2.3}$$

$$K_1(\tau,\ t) = \sqrt{\tau t}\int_0^\infty \{2\alpha\,[1 + \mathrm{th}\,\alpha h]^{-1} - s\}\,J_0(st)\,J_0(s\tau)\,ds.$$

Using the boundary condition (2.2), we obtain

$$\varphi_1(t,\ p) + \int_0^1 \varphi_1(\tau,\ p)\,K_2(\tau,\ t)\,d\tau = -\sqrt{t},\ 0 \leqslant t \leqslant 1, \tag{2.4}$$

$$K_2(\tau,\ t) = \sqrt{\tau t}\int_0^\infty \{2\alpha\,[1 + \mathrm{cth}\,\alpha h]^{-1} - s\}\,J_0(st)\,J_0(s\tau)\,ds,$$

where $\varphi_1(t, p)$ is defined by (1.11). The stress-intensity coefficient for a singularity at the nose of the crack is determined by the expression

$$K(p) = -P_1\varphi_1(1, p)\sqrt{l_0/2}, \tag{2.5}$$

where $\varphi_1(1, p)$ is the solution of (2.3), (2.4). For convenience of a calculation on an electronic computer, $K_i(\tau, t)$ (i=1, 2) can be taken in the form (1.16), where the values

$$\omega_1(\xi) = 2\sqrt{1+\xi^2}[1 + \text{th}\, ph\sqrt{1+\xi^2}]^{-1} - \xi - 2\xi/(4\xi^2+1),$$
$$\omega_2(\xi) = 2\sqrt{1+\xi^2}[1 + \text{cth}\, ph\sqrt{1+\xi^2}]^{-1} - \xi - 2\xi/(4\xi^2+1)$$

must be taken as $\omega_i(\xi)$.

The solid lines in Fig. 5 show the results of a numerical computation of the time dependence of the ratio $K(t)/(p_0\sqrt{l_0})$ by using the technique of finding the inverse Laplace transform numerically for $p(x, t) = p_0$. Curves 1 correspond to the ratio $l_0/h=1$, and curves 2, to $l_0/h=2$. The upper curves correspond to the boundary condition $\tau_{yz}=0$ for $y=-h$ and the lower, to $w=0$ for $y=-h$. Static solutions obtained from (2.3) and (2.4) as $p\to 0$ are represented in Fig. 6. The upper curve shows the change in the ratio $K/(p_0\sqrt{l_0})$ due to h/l_0 with the condition $\tau_{yz}=0$ at $y=-h$, and the lower curve corresponds to the condition $w=0$ at $y=-h$.

3. EXACT SOLUTIONS OF THE LIMIT PROBLEMS ($l_0 \gg h$)

Let us consider the problem of the dynamic loading of a semiinfinite crack located centrally in a layer of thickness 2h. Let us perform the same transformation to dimensionless quantities as (1.1) by replacing l_0 here by h. The crack is located at $y=0$ and $x<0$. Let us take $0<y<1$ as the domain under consideration. The boundary conditions of the problem are the following for $t > 0$:

$$\begin{gathered}\tau_{yz} = -p_0,\ y=0,\ x<0;\\ w=0,\ y=0,\ x>0;\\ \tau_{yz}=0,\ y=1,\ |x|<\infty.\end{gathered} \tag{3.1}$$

We can take

$$w=0,\ y=1,\ |x|<\infty \tag{3.2}$$

in place of the last condition in (3.1). In addition to the boundary conditions, the solution desired should satisfy additional conditions on the edge of the slit:

$$\begin{gathered}\tau_{yz} \sim x^{-1/2},\ x\to 0,\ x>0;\\ w \sim x^{1/2},\ x\to 0,\ x<0.\end{gathered} \tag{3.3}$$

After executing a Laplace integral transform in t and a Fourier transform in x, we obtain an ordinary differential equation for (1.2):

$$d^2w/dy^2 - (\lambda^2+p^2)w = 0,\ w = w(\lambda, y, p),$$

where $\lambda=\sigma+i\tau$ is a complex variable, and its general solution is

$$w(\lambda,y,p) = A(\lambda, p)\,\text{sh}\,\alpha y + B(\lambda, p)\,\text{ch}\,\alpha y,\ \alpha = \sqrt{\lambda^2+p^2}.$$

Using the boundary conditions (3.1), we obtain a Wiener–Hopf functional equation for the unknown functions τ_+ and w_-:

$$-\alpha\,\text{th}\,\alpha\cdot w_-(\lambda, p) = \tau_+(\lambda, p) + iP_0/(\lambda p),\ \ P_0 = (2\pi)^{-1/2}p_0, \tag{3.4}$$

where

$$\tau_+ = \tau_+(\lambda, p) = (2\pi)^{-1/2}\int_0^\infty \tau_{yz}(x, 0, p)\,e^{i\lambda x}dx;$$

$$w_- = w_-(\lambda, p) = (2\pi)^{-1/2}\int_{-\infty}^0 w(x, 0, p)\,e^{i\lambda x}dx.$$

Equation (3.4) is satisfied in the strip $-\gamma_0<\text{Im}\lambda<0$ ($\gamma_0>0$), $-\infty<\text{Re}\lambda<+\infty$ of the complex λ plane, where $\tau_+(\lambda, p)$ is a regular function in the domain $\text{Im}\lambda>-\gamma_0$, and $w_-(\lambda, p)$ is a regular function in the domain $\text{Im}\lambda<0$. Let us represent the function $K(\lambda)=\alpha\tanh\alpha$ as the product $K(\lambda)=K_+(\lambda)K_-(\lambda)$, where $K_+(\lambda)$ is a regular function without zeroes in the domain $\text{Im}\lambda>-\gamma_0$, and $K_-(\lambda)$, in the domain $\text{Im}\lambda<0$. Following [7], we obtain

$$K_+(-\lambda) = K_-(\lambda),$$
$$K_+(\lambda) = (p - i\lambda)\prod_{n=1}^{\infty} \frac{\sqrt{1 + p^2\pi^{-2}n^{-2}} - i\lambda(\pi n)^{-1}}{\sqrt{1 + p^2\pi^{-2}(n-1/2)^{-2}} - i\lambda\pi^{-1}(n-1/2)^{-1}}. \quad (3.5)$$

Using such a representation of $K(\lambda)$, we write (3.4) in the form

$$-w_-(\lambda, p)K_-(\lambda) - iP_0/p\cdot\chi_-(\lambda) = \tau_+(\lambda, p)K_+^{-1}(\lambda) + iP_0/p\cdot\chi_+(\lambda) = F(\lambda), \quad (3.6)$$

where

$$[\lambda K_+(\lambda)]^{-1} = \lambda^{-1}\left[K_+^{-1}(\lambda) - K_+^{-1}(0)\right] + \lambda^{-1}K_+^{-1}(0) = \chi_+(\lambda) + \chi_-(\lambda).$$

The left side of the equation is a function which is analytic in the domain $\mathrm{Im}\lambda < 0$, while the right side is analytic in the domain $\mathrm{Im}\lambda > -\gamma_0$. The function $F(\lambda)$ can be determined on the whole λ plane by analytic continuation, where $F(\lambda)$ will be regular in the whole λ plane.

Let us find the asymptotic of $K_+(\lambda)$ as $\lambda \to \infty$ and $\mathrm{Im}\lambda > 0$. To do this, let us compare the function $K_1(\lambda) = K_+^{-1}(\lambda)(p - i\lambda)$ at $\lambda = i\tau$ to the function

$$K_0(\tau) = \prod_{n=1}^{\infty} \frac{1 + \tau\pi^{-1}(n-1/2)^{-1}}{1 + \tau(\pi n)^{-1}} = \tau/\sqrt{\pi}\cdot\Gamma(\tau/\pi)\Gamma^{-1}(1/2 + \tau/\pi).$$

It can be shown that $\lim_{\tau\to\infty} K_1(\tau)\cdot K_0^{-1}(\tau) = 1$. Using the asymptotic of the gamma function, we obtain that $K_0(\tau) = \sqrt{\tau}$ as $\tau \to \infty$. It hence follows that

$$K_+(\lambda) = \sqrt{\tau} \quad \text{for } \lambda = i\tau,\ \tau \to \infty. \quad (3.7)$$

By using the relationship connecting the asymptotic of a function with the asymptotic of its Fourier transform [7], we obtain from the condition (3.3)

$$\tau_+(\lambda, p) \sim \lambda^{-1/2} \quad \text{as} \quad \lambda \to \infty,\ \mathrm{Im}\,\lambda > -\gamma_0;$$
$$w_-(\lambda, p) \sim \lambda^{-3/2} \quad \text{as} \quad \lambda \to \infty,\ \mathrm{Im}\,\lambda < 0. \quad (3.8)$$

The relationships (3.7) and (3.8) permit writing the following inequalities:

$$|-w_-(\lambda, p)K_-(\lambda) - iP_0/p\cdot\chi_-(\lambda)| < C|\lambda|^{-1};\ \mathrm{Im}\,\lambda < 0,\ |\lambda| \to \infty;$$
$$|\tau_+(\lambda, p)K_+^{-1}(\lambda) + iP_0/p\cdot\chi_+(\lambda)| < C|\lambda|^{-1};\ \mathrm{Im}\,\lambda > -\gamma_0,\ |\lambda| \to \infty,\ C = \text{const}.$$

Then according to the generalized Liouville theorem, the function $F(\lambda)$ from (3.6) equals zero, and therefore

$$w_-(\lambda, p) = -iP_0/p\cdot\chi_-(\lambda)K_-^{-1}(\lambda);\ \tau_+(\lambda, p) = -iP_0/p\cdot\chi_+(\lambda)K_+(\lambda).$$

Hence, by using (3.6) and (3.7), we obtain

$$\tau_+(\lambda, p) = -p_0/p\cdot\{\tau^{-1} - K_+^{-1}(0)\tau^{-1/2}\} \quad \text{for } \lambda = i\tau,\ \tau \to \infty. \quad (3.9)$$

Let us use the formulas connecting the asymptotic of a function to its Fourier transform [7]:

$$\tau(x) \sim Ax^{\eta},\quad x \to 0,\ x > 0;$$
$$\tau_+(\lambda) \sim A(2\pi)^{-1/2}\Gamma(1 + \eta)e^{\pi i(1+\eta)/2}\lambda^{-1-\eta},\ \lambda \to \infty.$$

As follows from (3.9), $\lambda = i\tau$, $\eta = -1/2$; hence, A equals the stress-intensity coefficient K(p) for a singularity on the order of $(\Delta x)^{-1/2}$ $(\Delta x \ll 1)$ at the nose of the crack

$$K_1(p) = p_0/p\cdot\pi^{-1/2}\sqrt{p\,\mathrm{cth}\,p}. \quad (3.10)$$

If (3.2) is taken in (3.1) in place of the last boundary condition, then the solution of the problem is carried out the same and the stress-intensity coefficient is hence

$$K_2(p) = p_0/p\cdot\pi^{-1/2}\sqrt{p\,\mathrm{th}\,p}. \quad (3.11)$$

The problem of dynamic loading of a semiinfinite crack parallel to the boundary of a half-space separated by the distance $h = 1$ is limiting when $l_0 \gg h$ for the problem considered in Sec. 2. The domain $-1 < y < \infty$ with a slit at $y = 0$, $x < 0$ is considered. As above, this domain is separated into two, the first with the subscript (1) $0 > y > -1$, and the second with the subscript (2) $y > 0$, $|x| < \infty$. The boundary conditions are the following for $t > 0$:

$$w_{(1)} = 0,\quad y = -1,\quad |x| < \infty;$$

$$\begin{aligned} &\tau_{(1)yz} = -p_0, && y=0, && x<0; \\ &\tau_{(1)yz} - \tau_{(2)yz} = 0, && y=0, && |x|<\infty; \\ &v = w_{(1)} - w_{(2)} = 0, && y=0, && x>0. \end{aligned} \quad (3.12)$$

We can take

$$\tau_{(1)yz} = 0, \; y = -1, \; |x| < \infty \quad (3.13)$$

in place of the first boundary condition. The Wiener–Hopf equation for the boundary-value problem (3.12) will be

$$\alpha[1 + \text{th}\,\alpha]^{-1} v_-(\lambda, p) = \tau_+(\lambda, p) + iP_0/(\lambda p).$$

The solution is carried out as above; hence,

$$K(\lambda) = \alpha[1 + \text{th}\,\alpha]^{-1} = K_+(\lambda)K_-(\lambda), \; K_+(-\lambda) = K_-(\lambda),$$

$$K_+(\lambda) = \sqrt{p - i\lambda}\, e^{-\varphi(\lambda) + i\alpha/\pi \cdot \ln[p^{-1}(-i\lambda + i\alpha)]} \prod_{n=1}^{\infty} \left[\sqrt{1 + p^2\pi^{-2}(n-1/2)^{-2}} - i\lambda\pi^{-1}(n-1/2)^{-1}\right] e^{i\lambda/\pi(n-1/2)};$$

$\varphi(\lambda) = -i\lambda/\pi \cdot \{1 - C + \ln[\pi/(2p)]\}$; $C = 0.5772\ldots$ is the Euler constant. The function $\varphi(\lambda)$ is defined in such a manner as to assure the algebraic order of the behavior of $K_+(\lambda)$ as $\lambda \to \infty$.

In this case we obtain the following expression for the stress-intensity coefficient:

$$K_4(p) = p_0 p^{-3/2}(2\pi)^{-1/2} e^{p/2}\, \text{ch}^{-1/2}\, p, \quad (3.14)$$

and by using the boundary condition (3.13),

$$K_3(p) = p_0 p^{-3/2}(2\pi)^{-1/2} e^{p/2}\, \text{sh}^{-1/2}\, p. \quad (3.15)$$

Using the relation between the asymptotic of a function and the asymptotic of its Laplace transform [7], we obtain from (3.10), (3.11) and (3.14), (3.15) that $K_1(t) = p_0 t/\sqrt{\pi}$, $K_2(t) = p_0/\sqrt{\pi}$, $K_3(t) = p_0 t/\sqrt{2\pi}$, $K_4(t) = p_0/\pi \cdot \sqrt{2t}$ as $t \to \infty$. Inverting the Laplace transforms in (3.10), (3.11) and (3.14), (3.15), we obtain

$$\begin{aligned} K_{1(2)}(t) &= (p_0/\pi)2\sqrt{t}\,[1 \pm H(t-2) + (1/2)H(t-4) \pm (1/2)H(t-6) + \ldots], \\ K_{3(4)}(t) &= (p_0/\pi)2\sqrt{t}\,[1 \pm (1/2)H(t-2) + (3/8)H(t-4) \pm (5/16)H(t-6) + \ldots], \end{aligned} \quad (3.16)$$

where $H(t-k) = \begin{cases} 1, & t>k \\ 0, & t<k \end{cases}$, and the lower signs in (3.16) correspond to $K_2(t)$ and $K_4(t)$. As is seen from the solutions obtained for $t \le 2$, i.e., although a reflected wave has still not arrived from the body boundary, they agree with the exact solution for a semiinfinite crack in an infinite elastic solid $K(t) = p_0/\pi \cdot 2\sqrt{t}$ [1].

The exact solutions (3.16) obtained are shown by the dashed lines 3 in Figs. 2 and 5 for $l_0 = 2h$. The solution for a semiinfinite crack in an elastic body is shown by the dashed lines 4 in Figs. 2 and 5. If the ratio between the maximum value of the dynamic and static stress-intensity coefficients is 1.27 [4], then as follows from a comparison between the curves corresponding to the dynamic loading with their appropriate curves for the static problems, it is seen that this ratio depends on the geometry of the problem and can be substantially larger.

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